Semi-symmetric non-metric s-Connection on a type of Hsu-Unified structure manifold

Dr. Shankar Kumar[1], Dr. Sundar Kumar Arya[2] and Prof. Jaya Upreti[3]

[1]Assistant Proffesor
Department Of Mathematics
S. S. J. V. S. S. S. Govt. Post Graduate College Ranikhet
Uttarakhand (India) , Pin- 263645

[2]Assistant Professor
Department Of Mathematics
L. S . M. Govt. Post Graduate College, Pithoragarh
Uttarakhand (India) , Pin- 262501

[3]Professor
Department Of Mathematics
S. S. J. Campus Almora
Uttarakhand (India), Pin-263601

Abstract - The study of semi-symmetric non-metric connection in a Riemannian manifold was intiated by Agashe and Chafle[1]. Later on Prasad, Kumar, Verma and De studied a types of semi-symmetric metric and non-metric connections on various types of manifolds [2,3,4,7,8,9,10].Ojha and Prasad[7] in 1986 studied semi-symmetric metric s-connection in a Sasakian manifold. In this paper we deal with a new class of Hsu-unified structure manifold M_n^* satisfying a certain condition. In section 2 we define semi-symmetric non-metric s-connection B. The Nijenhuis and Curvatue tensor along with their some interesting properties have also been studied in section 3 and 4. In section 5, we define Rcci tensor and scalar curvature in a new class of Hsu-unified structure manifold w.r.t. semi symmetric non metric s-connection B.

Keywords: Semi-symmetric non-metric s-Connection, Hsu-unified structure manifold, Nijenhius tensor and Curvature tensor.

1 Introduction

An even dimensional differentiable manifold M, n=2m of differentiability class C^∞, vector valued real linear function ϕ of differentiability class C^∞
satisfying

$$\phi^2 X = a^r X \tag{1}$$

also there exists a Riemannian metric g, such that

$$g(\overline{X},\overline{Y}) = a^r g(X,Y) \tag{2}$$

where $\overline{X} = \phi X, \quad 0 \leq r \leq n$
and
a is a real or complex number. X & Y are arbitrary vector fields.
Then in view of the equations (1) and (2), M is said to be a Hsu-unified structure manifold[6].
Let us define a symmetric 2-form F in M given as

$$F(X,Y) = g(\overline{X},Y) \tag{3}$$

$$F(X,Y) = F(Y,X) = g(\overline{X},Y) = g(X,\overline{Y}) \tag{4}$$

Then it is clear that the 2-form F satisfies

$$F(\overline{X},\overline{Y}) = a^r g(X,Y) \tag{5}$$

$$F(\overline{X},\overline{Y}) = a^r F(X,Y) \tag{6}$$

M is said to be a Hsu-Kahler manifold[5] if M satisfies the condition

$$(D_X\phi)Y = 0 \tag{7}$$

where D is Levi-Civita connection.
From the equation (7), we have

$$D_X\overline{Y} = \overline{D_X Y} \Leftrightarrow \overline{D_X\overline{Y}} = a^r(D_X Y) \tag{8}$$

Agreement 1.1

Here we define a special case of Hsu unified structure manifold, which stisfies the condition

$$D_X\xi = \overline{X} \tag{9}$$

and will be denoted by M_n^*

2 Semi Symmetric non metric s-connection *B*

If η is a vector field associated with ξ and ξ is 1-form, then an affine connection B which satisfies the condition

$$(B_X\phi)(Y) = \eta(Y)X - g(X,Y)\xi \tag{10}$$

is called s-connection[7].

B is called semi-symmetric non-metric connection[10] iff B satisfies

$$B_X Y = D_X Y - \eta(Y)X - g(X,Y)\xi \tag{11}$$

and

$$(B_X g)(Y,Z) = 2\eta(Y)g(X,Z) + 2\eta(Z)g(X,Y) \tag{12}$$

The torsion tensor S of M_n^* w.r.t. B is given by

$$S(X,Y) = B_X Y - B_Y X - [X,Y] \tag{13}$$

Where X and Y are vectors field.
With the help of equation (11), the equation (13) reduce to

$$S(X,Y) = \eta(X)Y - \eta(Y)X \tag{14}$$

We know that

$$g(X,\xi) = \eta(X) \tag{15}$$

now we have

$$\begin{aligned} B_X(\eta(Y) &= (B_X\eta)Y + \eta(B_X Y) \\ &= (B_X\eta)Y + D_X(\eta(Y)) - (D_X\eta)(Y) \\ &-\eta(X)\eta(Y) - g(X,Y)\eta(\xi) \end{aligned}$$

which implies

$$(B_X\eta)(Y) = (D_X\eta)Y + \eta(X)\eta(Y) + g(X,Y)\eta(\xi) \tag{16}$$

3 Nijenhuis tensor of M_n^* with respect to the connection B

The Nijenhuis tensor of ϕ in a hsu-unified structure manifold M_n^* is a vector valued bilinear function $\widetilde{N}(X,Y)$, defined by[6]

$$\widetilde{N}(X,Y) = (B_{\overline{X}}\phi)Y - (B_{\overline{Y}}\phi)X - \overline{(B_X\phi)Y} + \overline{(B_Y\phi)X} \tag{17}$$

where B is semi-symmetric non-metric s-Connection.

Theorem 3.1 *In Hsu-unified structure manifold M_n^*, Nijenhuis tensor vanishes with respect to the semi-symmetric non-metric s-connection B i.e.*

$$\widetilde{N}(X,Y) = 0 \tag{18}$$

Proof. From definition 3.1 we have

$$\widetilde{N}(X,Y) = (B_{\overline{X}}\phi)Y - (B_{\overline{Y}}\phi)X - \overline{(B_X\phi)Y} + \overline{(B_Y\phi)X}, \tag{19}$$

Replacing X by $\overline{X}$ in equation (10), we have

$$(B_{\overline{X}}\phi)Y = \eta(Y)\overline{X} - g(\overline{X}, Y)\xi, \tag{20}$$

interchanging X and Y in equation (20), we get

$$(B_{\overline{Y}}\phi)X = \eta(X)\overline{Y} - g(\overline{X}, Y)\xi, \tag{21}$$

operating ϕ on both side of equation (10), we get

$$\overline{(B_X\phi)Y} = \eta(Y)\overline{X} - g(X, Y)\phi \circ \xi, \tag{22}$$

interchanging X and Y in equation (22)

$$\overline{(B_Y\phi)X} = \eta(X)\overline{Y} - g(X, Y)\phi \circ \xi, \tag{23}$$

using equations (20), (21), (22) and (23) in equation (19), we get equation (18).

4 Curvature tensor $\tilde{R}(X, Y, Z)$ of M_n^* with respect to the connection B

The curvature tensor $\tilde{R}(X, Y, Z)$ of M_n^* with respect to semi symmetric non metric s-connection B, Analogous to the definition of the curvature tensor with w.r.t. Levi-Civita connection D, given by[6]

$$\tilde{R}(X, Y, Z) = B_X B_Y Z - B_Y B_X Z - B_{[X,Y]}Z \tag{24}$$

In this section we have following theorems:

Theorem 4.1 *In hsu-unified structure manifold M_n^* equipped with a semi-symmetric non metric s-connection B, the curvature tensor $\tilde{R}(X, Y, Z)$ is given by*

$$\begin{aligned}\tilde{R}(X, Y, Z) = R(X, Y, Z) - \beta(X, Z)Y + \beta(Y, Z)X \\ -g(Y, Z)PX + g(X, Z)PY\end{aligned} \tag{25}$$

where $R(X, Y, Z)$ is curvature tensor of M_n^* with respect to the Riemannian connection D, given as

$$R(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$$

β is a tensor field of type (0,2) defined as

$$\beta(X, Y) = (D_X\eta)Y + \eta(X)\eta(Y) + g(X, Y)\eta(\xi), \tag{26}$$

P is defined as

$$PX = (D_X\xi - \eta(X)\xi) \tag{27}$$

Proof. Using equation (11) in (24) and with the help of equations (26) & (27), we can easily get equation (25).

Theorem 4.2 *The tensor field of type* $(0,2)$ *defind by equation* (26) *is commutative iff* a *is closed 1-form. where* d *is defined as*

$$dn(X,Y) = (D_X\eta)Y - (D_Y\eta)X \tag{28}$$

Proof. In consequence of equation (26), we have

$$\beta(X,Y) - \beta(Y,X) = dn(X,Y). \tag{29}$$

If a is closed 1-form, then from equation (29), we get

$$dn(X,Y) = (D_X\eta)Y - (D_Y\eta)X = 0 \tag{30}$$

Using equation (30) in equation (29), we have

$$\beta(X,Y) = \beta(Y,X) \tag{31}$$

Definition

Let K and $\widetilde{K}$ be the curvature tensors of type $(0,4)$ w.r.t Riemannian connection D and s-connection B given as[6]

$$K(X,Y,Z,U) = g(R(X,Y,Z),U), \tag{32}$$

and

$$\widetilde{K}(X,Y,Z,U) = g(R(X,Y,Z),U). \tag{33}$$

In this section we have the following theorems:

Theorem 4.3 *In hsu-unified structure manifold* M_n^* *with semi-symmetric non-metric s-connection* B *with closed 1-form* a*, we have*

$$\tilde{R}(X,Y,Z) + \tilde{R}(Y,Z,X) + \tilde{R}(Z,X,Y) = 0 \tag{34}$$

$$\widetilde{K}(X,Y,Z,U) + \widetilde{K}(Y,X,Z,U) = 0 \tag{35}$$

Proof. Using equations (25),(31) & Bianchi,s first identity of curvature tensor with respect to Levi-Civita connection D, i.e.

$$R(X,Y,Z) + R(Y,Z,X) + R(Z,X,Y) = 0 \tag{36}$$

we can easily find equation (34).

From equations (32) and (33) we have

$$\begin{aligned}\tilde{K}(X,Y,Z,U) = K(X,Y,Z,U) - [(D_X\eta)Z]g(Y,U) \\ -\eta(X)\eta(Z)g(Y,U) - \eta(\xi)g(X,Z)g(Y,U) \\ +[(D_Y\eta)Z]g(X,U) + \eta(Y)\eta(Z)g(X,U) \\ +\eta(\xi)g(Y,Z)g(X,U) - g(Y,Z)g(D_X\xi,U) \\ +\eta(X)\eta(U)g(Y,Z) + g(X,Z)g(D_Y\xi,U) \\ -\eta(Y)\eta(U)g(X,Z)\end{aligned} \quad (37)$$

We know that

$$K(X,Y,Z,U) = -K(Y,X,Z,U) \quad (38)$$

Using equations (38) & (9) we get equation (36).

5 Ricci tensor $\tilde{S}$ and scalar curvature $\tilde{r}$ of M_n^* with respect to s-Connection B

Let M_n^* be a special case of n-dimensional Hsu-unified structure manifold defined in equation (9). Then the Ricci tensor $\tilde{S}$ of the manifold M_n^* with respect to the semi-symmetric non-metric s-connection B is defined by [6]

$$\tilde{S}(X,Y) = \sum_{i=1}^{n} \varepsilon_i g(\tilde{R}(e_i,X,Y),e_i) \quad (39)$$

and the scalar curvature of the manifold M_n^* with respect to the connection B is given by

$$\tilde{r} = \sum_{i=1}^{n} \varepsilon_i \tilde{S}(e_i,e_j) \quad (40)$$

where $\{e_1,e_2,\ldots\ldots\ldots,e_n\}$is an orthonormal frame and $\varepsilon_i = g(e_i,e_j)$.

Theorem 5.1 *In M_n^*, the Ricci tensor $\tilde{S}$ and scalar curvature $\tilde{r}$ of connection B are given by*

$$\begin{aligned}\tilde{S}(Y,Z) = S(Y,Z) + (n-1)\beta(Y,Z) + a^r g(Y,Z) \\ +g(\overline{Y},Z) - \eta(Y)\eta(Z)\end{aligned} \quad (41)$$

and

$$\tilde{r} = r + a^r(n-1)(n+2) \quad (42)$$

Proof. By using equation (25) in equation (39), we easily find equation (41) and result shown by (42) is obvious from equation (41).

References

[1] Agashe N.S. and Chafle M. R., A semi symmetric non metric connection in a riemannian manifold; Indian

J. pure appl. math. 23, 399-409 (1992).

[2] De, U. C. and Biswas, S. S., On a type of semi-symmetric metric connection on a Riemannian manifold, Publ. de. L' Institute Math., 61, 90-96(1969).

[3] De, U. C. and De, B. K., On a type of semi-symmetric connection on a Riemannian manifold, Ganita, 47(2), 11-14 (1996).

[4] De U. C. and J. Sengupta, On a type of semi symmetric metric connection on a almost contact manifold; Bull. cal. math. soc. 95(5),375-384 (2000)

[5] Kumar S.,Kandpal, D. and Upreti, J., On a hsu unified structure manifold with a recurrent metric connection; Comp. & Math. Sci. Vol.8 (8), 366-372 (2017).

[6] Mishra, R. S., Structures on a differentiable manifold and their applications, Chandrama Prakashan, Allahabad, India, 1984.

[7] Ojha, R. H. and Prasad, S. S., On semi-symmetric s-connection in a Sasakian manifold; Ind. Journal Pure and Appl. Math., 16(4), 341-344 (1986).

[8] Prasad, B., On a semi-symmetric non-metric connection in SP-sasakian manifold,Istanbul Univ. Fen. Fak. Math. Der., 53, 77-80 (1994).

[9] Prasad, B. and Kumar, R., A semi-symmetric Non-metric sp-connection in an SP-Sasakian manifold, Jour. Nat. Acad. Math., 16, 37-45 (2002).

[10] Prasad, B. and Verma, R. K., On a type of semi-symmetric non-metric connection on a Riemannian manifold, Bull. Cal. Math. Soc., 96(6), 483-488 (2004).

[11] Prasad, B., Verma, R. K. and De, U. C., On a type of semi-symmetric connection on a Riemannian manifold, Bull. Cal. Math. Soc., 100(3), 227-240.(2008)